The Links Between Smale's Mean Value Conjecture and Complex Dynamics

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Abstract

In this paper we study the links between Smale's Mean Value Conjecture (SMVC) and the convergence of critical points under iteration. We begin by introducing SMVC and discussing what progress has been made towards proving this conjecture. From there we give a brief introduction to a few concepts in Complex Dynamics and Complex Analysis including conjugacy, conjugations by $1/z$, and orbits. We then show that conjugacy conserves SMVC. At this point, we go through the quadratic case to show the patterns that we are looking for as well as to show a basic structure of how this problem is being looked at. From there we look at the cubic case, going through SMVC for the cubic polynomial. We then introduce the Petal Theorem and show how it is used for the cubic case specifically. We conclude with the connections between the SMVC in the cubic case and the convergence of the same critical points explained and discuss future work.

1 Introduction and Background

Complex Dynamics is the study of complex functions under iteration. We are specifically interested in where certain points in the complex plane converge to the origin and to infinity. It is this idea that we kept in mind when we looked at Smale's Mean Value Conjecture (SMVC) which states [7]:

Conjecture 1.1. *Let $f(z) = z + \sum_{i=2}^{n} a_i z^i$ be a complex valued polynomial of degree $n \geq 2$ for which $f(0) = 0$ and $f'(0) = 1$. Then, there exists a critical point c of this polynomial that satisfies the following statement:*

$$|\frac{f(c)}{c}| \leq 1$$

A variation of this conjecture has been announced for all polynomials of degree 10 or less by Sendov and Marinov [6]. This variation replaces 1 with an even better bound of $(d-1)/d$. However, their paper has no proofs to show this result. Several other papers use another variation on this conjecture. Namely, this following proposition:

Proposition 1.2. *Let p be a polynomial of degree $N \geq 2$ over $\mathbb{C}$, and supposed that $x \in \mathbb{C}$ is not a critical point of p. Then there exists a critical point ζ of p such that*

$$|\frac{p(\zeta) - p(x)}{\zeta - x}| \leq 4|p'(x)|. \quad (1)$$

In a paper written by E. Crane [2], he shows that there can be a better bound than the bound of 4 that was proven by S. Smale in 1981 [7] in the proposition given above. He proves that for all polynomials of degree 8 and higher, $K(d) < 4 - \frac{2.263}{\sqrt{d}}$ where d is the degree of the polynomial and $K(d) = sup\{S(p,x) : deg(p) = d, p'(x) \neq 0\}$ $(S(p,x) = min(|\frac{p(\zeta)-p(x)}{(\zeta-x)p'(x)}| : p'(\zeta) = 0))$. For $2 \leq d \leq 7$, Schmeisser [5] proved that we can have $K(d) \leq \frac{2^d-(d+1)}{d(d-1)}$ which is a better bound than what Crane would have for those values of d. In 2003, T.W. Ng proved that this original bound could be further reduced to 2 for all nonlinear odd polynomials with a nonzero linear term [3].

Although there have been many papers published on the possibility of this conjecture (and its variations) being true, none have been able to prove it completely. The goal of this paper is not to prove the conjecture, however. This paper will look into possible connections between SMVC and Complex Dynamics. SMVC says nothing about which critical point or points satisfy the bound. It only states that there is one. On the other hand, complex dynamics says that given a polynomial $f(z) = z + \sum_{i=2}^{n} a_i z^1$, there must exist a critical point tending to the origin under iteration (this fact is presented formally in Proposition 1.8). We hope to combine these two thoughts to see if there is a relation between them. Specifically, we will prove the following main theorem:

Theorem 1.3. *For any cubic of the form $p(z) = z + a_2z^2 + a_3z^3$, there is a critical point satisfying the SMVC bound and converging to the origin.*

To begin with, we will be wanting to show that this works in general. It is generally nice to be able to work with one form instead of many so to do this we will use the idea of conjugacy:

Definition 1.4. If there exists a function $\phi(z)$ that is either a linear fractional transformation (such as $1/z$) or a so called afine map (such as $Az + B$), then two functions are conjugate if $\phi \circ f = g \circ \phi$.

Below is an example of this concept:

Example 1.5. Let $f(z) = z^2 - z$ and $\phi(z) = \alpha z + \beta$. We want to find out if this form is conjugate to $g(w) = w^2 + c$ where c is some constant. To find out if these two are conjugate, we will follow this chart:

$$\begin{array}{ccc} Z & \xrightarrow{z^2-z} & \mathbb{C} \\ \phi\downarrow & & \downarrow\phi \\ W & \xrightarrow{w^2+c} & \mathbb{C} \end{array}$$

Following the chart, we see that we get $\alpha(z^2 - z) + \beta = (\alpha z + \beta)^2 + c \Rightarrow \alpha z^2 - \alpha z + \beta = \alpha^2 z^2 + 2\alpha\beta z + \beta^2 + c$. For two polynomials to be equal to one another, all of the coefficients must equal one another. With this in mind, we find that $\alpha = \alpha^2, -\alpha = 2\alpha\beta, \beta = \beta^2 + c \Rightarrow \alpha = 1, \beta = -1/2, c = -3/4$. Since there exists an α and β that makes this true, we know that these two polynomials are conjugate for $c = -3/4$. This means that if we let $\phi(z) = \alpha z + \beta$, we have that $\phi \circ f = g \circ \phi$.

From here, we need to show that we can use the conjugated form to look at SMVC. Below is a lemma that shows this for $\phi(z) = \alpha z$.

Lemma 1.6. *If $p(z) = z + h.o.t.$ satisfies SMVC and if $q = A \circ p \circ A^{-1}$ where $A(z) = \alpha z \, (\alpha \neq 0)$, then q also satisfies SMVC.*

Proof. Suppose that c is a critical point of $p(z)$ and that it satisfies $|p(c)/c| \leq 1$. Since they are conjugate via multiplication by α, the point $x = \alpha c$ is a critical point of q. Since $q = A \circ p \circ A^{-1}$ we can see that $|q(x)/x| = |q(\alpha c)/(\alpha c)| = |\frac{\alpha(p((\alpha c)/\alpha))}{\alpha c}| = |p(c)/c| \leq 1$. Hence, if $p(z)$ satisfies SMVC and if $q = A \circ p \circ A^{-1}$ where $A(z) = \alpha z \, (\alpha \neq 0)$, then $q(z)$ will also satisfy SMVC. $\square$

At this point, I need to introduce the concept of an orbit and a general proposition that uses this concept as well as talk about the conjugacy of $1/z$ which will be used in the cubic case.

Definition 1.7. Suppose X is any set and $f : X \mapsto X$ is any function. Let $x_0 \in X$. Then, the orbit of x_0 is defined as x_0, $f(x_0)$, $f(f(x_0))$, ...

This concept is used in the following proposition:

Proposition 1.8. *There exists a critical point c_i such that the orbit of c_i converges to the origin.*

From here, I will introduce the conjugacy of $1/z$. In general, $1/z$ maps "circles" to "circles." When I say "circles" I mean that they can fit into this general form: $A(x^2 + y^2) + Bx + Cy + D = 0$. At this point, we want to see what x and y are equal to in terms of the new coordinates u and v. After some computations, we see that $x = \frac{u}{u^2+v^2}$ and $y = \frac{-v}{u^2+v^2}$. After we use these to substitute into the general form, we find that in the new plane we get $D(u^2 + v^2) + Bu - Cv + A = 0$. So, lines become circles, circles become circles, and circles can sometimes become lines. If we are mapping a circle that is around the origin, then everything inside goes outside and vice versa. This is because the origin goes to ∞ and ∞ goes to 0. This is the method that I will use in the future.

Now that we have a basic background in Complex Analysis as well as Complex Dynamics, we can look into the quadratic case as an example.

2 The Quadratic Case

To begin with, we will go through the quadratic case. The following Theorem describes what we are planning to prove in this section:

Theorem 2.1. *For each quadratic polynomial $f(z) = z + a_2 z^2$, there is exactly one critical point. It satisfies the bound in SMVC and it converges to the origin*

Proof. The form that we use is a very general form of the quadratic polynomials. However, we want to use a simpler version by conjugating this polynomial by $\phi(z) = \alpha z$ using the process explained in the previous section.

$$\begin{array}{ccc} Z & \xrightarrow{z+a_2z^2} & \mathbb{C}_\infty \\ \phi(z)\downarrow & & \downarrow\phi(z) \\ Z' & \xrightarrow{z-z^2} & \mathbb{C}_\infty \end{array}$$

This means that we need $\alpha(z+a_2z^2) = \alpha z - \alpha^2 z^2 \Rightarrow z + a_2z^2 = z - \alpha z^2 \Rightarrow \alpha = -a_2$. So, we can now do all of our calculations using $f(z) = z - z^2$. First, we need to find its critical point: $f'(z) = 1 - 2z = 0 \Rightarrow z = 1/2$. Referring back to SMVC, we see that $|\frac{f(1/2)}{1/2}| = |\frac{1}{4}\cdot 2| = |\frac{1}{2}| \leq 1$. Hence, SMVC is satisfied by all quadratic polynomials of this form.

Now, we want to look at where this critical point converges to the origin. To get an idea of this, we will refer back to Proposition 1.8. In this case, $i = 1$. So, since there is only one critical point, this point must converge to the origin. Hence, this one critical point both satisfies SMVC and converges to the origin. □

At this point, we want to see if this pattern continues for the cubic case.

3 The Cubic Case

In this section, we plan to prove the main theorem (Theorem 1.3). We will do this both indirectly and directly. The indirect approach will use Proposition 1.8 and handles "areas" or "regions." The direct approach will use Beardon's Petal Theorem which will be introduced later on. To begin with, we will be looking at the following general form of cubic polynomials that satisfy the conditions in SMVC:

$$p(z) = z + az^2 + bz^3 \tag{2}$$

However, we would like this in a more useful form. To do this, we will conjugate this polynomial by $\phi(z) = \alpha z$:

$$\begin{array}{ccc} Z & \xrightarrow{z+az^2+bz^3} & \mathbb{C} \\ \phi\downarrow & & \downarrow\phi \\ Z & \xrightarrow{z-\frac{1}{2}(c+\frac{1}{c})z^2+\frac{1}{3}z^3} & \mathbb{C} \end{array}$$

Following the same process described in example 1.5, we see that we have $z + az^2 + bz^3 = z - \frac{\alpha}{2}(c + \frac{1}{c})z^2 + \frac{\alpha^2}{3}z^3$. For two polynomials to be equal, their coefficients must be equal: $a = -\frac{\alpha}{2}(c + 1/c)$ and $b = \frac{\alpha^2}{3}$. Hence, $\alpha = -\frac{2a}{c+1/c}$. Since there exists an α we know that these two polynomials are conjugate and we can continue with the following cubic:

$$f_c(z) = z - \frac{1}{2}(c + \frac{1}{c})z^2 + \frac{1}{3}z^3 \tag{3}$$

After taking the derivative, it is easy to see the two critical points: $c_1 = c$ and $c_2 = 1/c$. These two critical points are nice to work with since they create the same map. In other words, $f_c(z) = f_{1/c}(z)$. This means that by looking at one of the critical points, we have a way of looking at the other. For example, if we can show for one critical point that $f_c^{on}(c_1(c)) \to \infty \Leftrightarrow f_{1/c}^{on}(c_1(c)) \to \infty \Leftrightarrow f_{1/c}^{on}(c_2(1/c)) \to \infty$ then we see that in an area that is $1/c$ we have the same thing for the other critical point. So, we will look at what happens to c_1 to get an idea what happens to c_2 as well. At this point, we need to look at where c_1 and c_2 satisfy SMVC:

$$f_c(c) = c - \frac{1}{2}(c+\frac{1}{c})c^2 + \frac{1}{3}c^3 = \frac{3c - c^3}{6} \Rightarrow |\frac{f_c(c)}{c}| = |\frac{3-c^2}{6}| \leq 1 \Rightarrow |3-c^2| \leq 6$$

$$f_c(\frac{1}{c}) = \frac{1}{c} - \frac{1}{2}(c+\frac{1}{c})\frac{1}{c^2} + \frac{1}{3c^3} = \frac{3c^2-1}{6c^3} \Rightarrow |\frac{f_c(\frac{1}{c})}{\frac{1}{c}}| = |\frac{3c^2-1}{6c^2}| \leq 1 \Rightarrow |3 - \frac{1}{c^2}| \leq 6$$

I claim that with these two graphs combined, we see that SMVC is satisfied for any c throughout the parameter plane. Below is a figure that shows this fact.

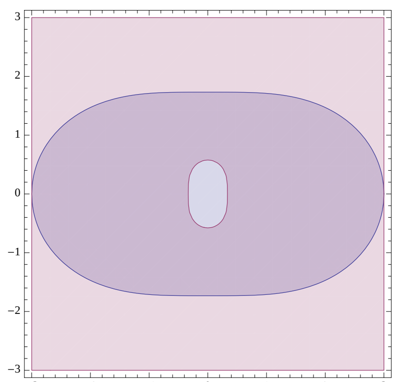

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Figure 1: The smallest area is where only c_1 satisfies SMVC, the darkest area shows where both critical points satisfy SMVC, and the area outside the larger curve shows where only c_2 satisfies SMVC. Put together, they cover the entire complex plane.

At this point, we want to look at areas of convergence to infinity. For $c_1 = c$, we will prove the following lemma to show where it converges to infinity:

Lemma 3.1. *Let $\delta = \frac{3}{4}\sqrt{30}$. If $|c| > \delta$ and $|z| > |c|^3/6$ then $|f_c(z)| > |z|$ and, under iteration of f, z converges to infinity.*

Proof. We know that $|f_c(z)| = |z||1 - \frac{1}{2}(c+1/c)z + \frac{1}{3}z^2|$ and that we want $|z||1 - \frac{1}{2}(c+1/c)z + \frac{1}{3}z^2| > |z|$. So, proving that $|1 - \frac{1}{2}(c+1/c)z + \frac{1}{3}z^2| > 1$ will imply the previous inequality. However, if we prove that $|-\frac{1}{2}(c+1/c)z + \frac{1}{3}z^2| > 2$ then the previous inequality will be true. We can rewrite this as $|z||-\frac{1}{2}(c+1/c) + \frac{1}{3}z| > 2$, which can be implied by the following: $|-\frac{1}{2}(c+1/c) + \frac{1}{3}z| > 2/|z| > 2/(|c|^3/6) >$

0.17. Using the triangle inequality we see that $|-\frac{1}{2}(c+1/c)+\frac{1}{3}z| \geq ||\frac{1}{2}(c+1/c)|-\frac{1}{3}|z|| = |\frac{1}{3}|z|-|\frac{1}{2}(c+1/c)|| > |\frac{|c|^3}{18}-|\frac{1}{2}(c+1/c)|| \geq |\frac{|c|^3}{18}-\frac{1}{2}(|c|+|1/c|)|$. This function is monotonically increasing as a function of $|c|$. So, we can re-write it as $\frac{|c|^3}{18}-\frac{1}{2}(|c|+|1/c|) > 1.67$. Since $1.67 > 0.17$ we see that all of the previous inequalities hold. Hence, under iteration of f, z converges to infinity. $\square$

This means that for all values of $|c|$ greater than $\frac{3\sqrt{30}}{4}$, c_1 will converge to infinity. Based on the earlier explanation, we know that c_2 will also converge to infinity in the circle of $|c| > \frac{3}{4}\sqrt{30}$ mapped by $1/c$ in the parameter plane. This will give us the area where c_2 converges to infinity. This circle will be $|c| < \frac{4}{3\sqrt{30}}$. Now that we have two areas where c_1 and c_2 converge to infinity, we want to refer back to Proposition 1.8. In this case, $i = 2$. This means that, for example, in the area where c_1 converges to infinity, we know that c_2 must converge to the origin under iteration. The same happens in the area where c_2 converges to infinity. From here we can combine these circles with Figure 1 to create five distinct areas:

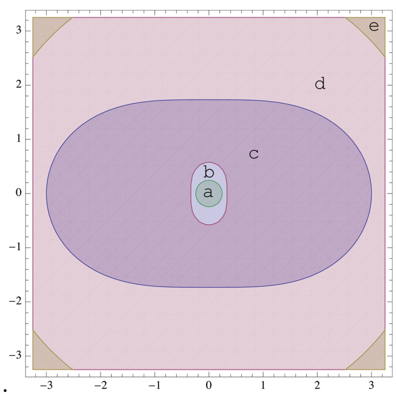

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Figure 2: There are five different points representing the five different areas created by four graphs.

We are interested in what happens within each of those areas. Below are five charts that give us an idea of what is going on:

Region a	c_1	c_2
SMVC	y	n
$\to 0$	y	n

Region b	c_1	c_2
SMVC	y	n
$\to 0$	?	?

Region c	c_1	c_2
SMVC	y	y
$\to 0$	?	?

Region d	c_1	c_2
SMVC	n	y
$\to 0$	?	?

Region e	c_1	c_2
SMVC	n	y
$\to 0$	n	y

As you can see, only regions a and e have full charts. We can see that each of those have a column of yeses. This is the pattern that we are looking for.

This means that the chart for region c does satisfy this pattern automatically based on Proposition 1.8. Since in this area both c_1 and c_2 satisfy SMVC and we know that one of those critical points will converge to the origin under iteration, we know that we will have a column of yeses. Hence, the two regions that we are interested in are b and d. However, since our critical points have the same map, we can study the area that contains point d and be able to have an understanding of the area that contains point b. To be able to get a good understanding of what is happening in region d, we will need to look at the annulus $1.75 \leq |c| \leq 4.11$ which encompasses the entire region of d. In this annulus, we will be looking at the following theorem:

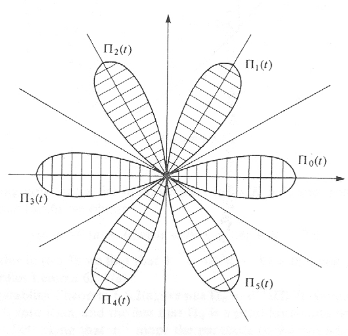

Figure 3: This shows what is meant in the following theorem by "petal." In our specific case, we will only have one petal.

Theorem 3.2. *(The Petal Theorem) Suppose that the analytic map f has a Taylor Expansion*

$$f(z) = z - z^{p+1} + O(z^{2p+1}) \tag{4}$$

at the origin. Then for all sufficiently small t:

1. *f maps each petal $\Pi_k(t)$ into itself;*
2. *$f^n(z) \to 0$ uniformly on each petal as $n \to \infty$;*
3. *$argf^n(z) \to 2k\pi/p$ locally uniformly on $\Pi_k asn \to \infty$;*
4. *$|f(z)| < |z|$ on a neighborhood of the axis of each petal;*
5. *$f : \Pi_k(t) \to \Pi_k(t)$ is conjugate to a translation. [1]*

Since our $f_c(z)$ is not in the form that is needed to be able to use this theorem, we will need to conjugate it to create the form that is needed:

$$\begin{array}{ccc} Z & \xrightarrow{z-\frac{1}{2}(c+\frac{1}{c})z^2+\frac{1}{3}z^3} & \mathbb{C} \\ \phi\downarrow & & \downarrow\phi \\ Z & \xrightarrow{z-z^2+c'z^3} & \mathbb{C} \end{array}$$

where $\phi(z) = \alpha z$. This shows that we need to find what α is as well as c':

$$\alpha(z-\frac{1}{2}(c+\frac{1}{c})z^2+\frac{1}{3}z^3) = \alpha z-\alpha^2z^2+c'\alpha^3z^3 \Rightarrow z-\frac{1}{2}(c+\frac{1}{c})z^2+\frac{1}{3}z^3 = z-\alpha z^2+c''\alpha^2z^3$$

So, for two polynomials to be equal, their coefficients must be equal. This means that $\alpha = \frac{1}{2}(c+\frac{1}{c})$ and that $\frac{1}{3} = c'(\frac{1}{2}(c+\frac{1}{c}))^2$. So, we now know that $\alpha = \frac{1}{2}(c+\frac{1}{c})$ and $c' = \frac{1}{3}(\frac{2c}{c^2+1})^2$. Since we are now working in a new plane, which we will call the u-plane, we need to convert our annulus into the new coordinates. The following lemma shows this:

Lemma 3.3. *Let $R = \{c : 1.75 \leq |c| \leq 0.96\}$ and let $R' = \{c' : 0.07 < |c'| < 0.96\}$. Then $c \in R \Rightarrow c' \in R'$.*

Proof. To show this, we need to reduce c' to a more useful form: $|c'| = |\frac{1}{3}(\frac{c2}{c^2+1})^2| = \frac{4}{3}|\frac{1}{c+1/c}|^2$. To begin with, we want to look at the denominator of the main fraction. Using the triangle inequality, we see that $|c+\frac{1}{c}| \geq ||c|-\frac{1}{|c|}|$. To be able to get a better bound on this, we need to show that $|c|-\frac{1}{|c|}$ is monotonically increasing as a function of $|c|$. To do this, we want to make sure that the derivative is always positive for values between 1.75 and 4.11: $1+\frac{1}{|c|^2}$ is always positive. This means that the smallest value will happen at the bound. So, we have that $||c|-\frac{1}{|c|} \geq |\frac{7}{4}-\frac{4}{7}| = |\frac{33}{28}|$. This means that $\frac{1}{|c+1/c|} \leq |\frac{28}{33}|$. Combining this with the rest of the function, we get an upper bound: $|c'| \leq \frac{4}{3}|\frac{28}{33}|^2 < 0.96$.

For the lower bound, we will follow a similar process. By the triangle inequality, we see that $|c+\frac{1}{c}| \leq |c|+\frac{1}{|c|}$. Before we can continue, we need to show that this function is monotonically increasing as a function of $|c|$. The derivative is $1-\frac{1}{|c|^2}$ which is always positive between 1.75 and 4.11. This means that the greatest value will happen at the boundary. So, we have that $|c|+\frac{1}{|c|} \leq 4.11+\frac{1}{4.11} < 4.35$. Putting this into the rest of the function, we find that the lower bound is $|c'| > 0.07$. □

Now that we have bounds in our new coordinate system, we want to go back to the Petal Theorem. We will show that this theorem works for our case which means that $p = 1$ and $k = 0$ (assuming that we are ignoring the points $c = \pm i$). To do this, we want to prove the following lemma:

Lemma 3.4. *Given z in the dynamical plane of f_c, let $z' = (1/2)(c+1/c)z$ be the corresponding point in the dynamical plane, and put $w = 1/z'$. If $c \in R$ and $Re(w) > 4.36$ then under iteration of f_c, the point z converges to the origin.*

Proof. To begin with, we want to conjugate $p(z) = z - z^2 + c'z^3$ to the w-plane by the conjugation $\sigma : z \mapsto 1/z$. First, we will look at what $1/p(z)$ is. Using polynomial division, we find:

$$\frac{1}{z - z^2 + c'z^3} = \frac{1}{z} + 1 + Az + \nu(z), \tag{5}$$

where $A = 1 - c'$ and $|\nu(z)| = |\frac{(1-2c')z^2 - c'(1-c')z^3}{1-z+c'z^2}| \leq B|z|^2, B > 0$.

At this point we need to find bounds on both B and A by proving this corollary:

Corollary 3.5. *If $c' \in R'$ and $|z| < 0.23$ then $|A| < 1.96$ and $|\nu(z)| \leq B|z|^2$ where $B = 4.66$.*

Proof. First, we will look at finding a bound on A. We know that $A = 1 - c'$ so $|A| = |1 - c'|$. Using the triangle inequality, we see that $|A| = |1 - c'| \leq 1 + |c'|$. Since $c' \in R'$, we know that $|c'| < 0.96$. So, $1 + |c'| < 1.96$ and hence $|A| < 1.96$.

Now we want to find B and so we need a bound on $\nu(z)$. By definition, $|z| < 0.23$. This choice will be explained later. First, we will look at the denominator of $\nu(z)$. By the triangle inequality, we see that $|1 - z + c'z^2| \geq ||1 - z| - |c'z^2|| = |1 - z| - |c'||z|^2 \geq 1 - |z| - |c'||z|^2 \geq 1 - 0.23 - (0.96)(0.23)^2 \approx 0.719$.

Now that we have a bound on the denominator of $\nu(z)$, we can continue to get a bound on the full function which will become our B. To do this, let's first look at the numerator separately: $|(1 - 2c')z^2 - c'(1 - c')z^3| = |z|^2 \cdot |(1 - 2c') - c'(1 - c')z|$. Since the bound on $|\nu(z)|$ is $B|z|^2$, we can disregard the $|z|^2$ term. Using the triangle inequality and the fact that $|z| < 0.23$ we see that $|(1 - 2c') - c'(1 - c')z| \leq |1 - 2c'| + |c'(1 - c')||z| < |1 - 2c'| + 0.23|c'(1 - c')|$. Earlier we were able to find a bound on the denominator, so if we let $B = \frac{|1-2c'|+0.23|c'(1-c')|}{0.719}$, we can guarantee that $|\nu(z)| \leq B|z|^2$.

At this point,we want to find a bound on B. Since $B = \frac{|1-2c'|+0.23|c'(1-c')|}{0.719}$, we know by the triangle inequality that $\frac{|1-2c'|+0.23|c'(1-c')|}{0.719} \leq \frac{1+2|c'|+0.23|c'|(1-|c'|)}{0.719}$. Using the fact that $|c'| < 0.96$ since $c' \in R'$, we have that $\frac{1+2|c'|+0.23|c'|(1-|c'|)}{0.719} < 4.66$. □

At this point we want to complete the proof by showing the following corollary:

Corollary 3.6. *Given c in R, the critical point $c_2 = 1/c$ converges to the origin.*

Proof. I claim that we can show that c_2 converges by using the second iterate. If we can show that the second iterate converges to the origin, then we can conclude that c_2 converges to the origin as well. Recall that $f_c(c_2) = \frac{3c^2-1}{6c^3}$. Iterating one more time, we find that

$$f_c(f_c(c_2)) = \frac{6c(6c^3)^2(3c^2 - 1) - 3c^2(6c^3)(3c^2 - 1)^2 - 3(6c^3)(3c^2 - 1)^2 + 2c(3c^2 - 1)^3}{6c(6c^3)^3}.$$

We need to convert this into the new coordinate system. To do this, multiply the above function by $\alpha = \frac{c^2+1}{2c}$. After working through the computation, the result is $\frac{5}{48c^2} - \frac{5}{72c^4} + \frac{1}{72c^6} - \frac{1}{1296c^8} - \frac{1}{1296c^{10}} + \frac{3}{16}$. From here, we want an upper bound on $\frac{5}{48c^2} - \frac{5}{72c^4} + \frac{1}{72c^6} - \frac{1}{1296c^8} - \frac{1}{1296c^{10}}$ to create a circle whose radius is that upper bound, centered at $\frac{3}{16}$. To get an upper bound on this, we will use the triangle inequality: $|\frac{5}{48c^2} - \frac{5}{72c^4} + \frac{1}{72c^6} - \frac{1}{1296c^8} - \frac{1}{1296c^{10}}| \leq |\frac{5}{48c^2}| + |\frac{5}{72c^4}| + |\frac{1}{72c^6}| + |\frac{1}{1296c^8}| + |\frac{1}{1296c^{10}}| \leq |\frac{5}{48(7/4)^2}| + |\frac{5}{72(7/4)^4}| + |\frac{1}{72(7/4)^6}| + |\frac{1}{1296(7/4)^8}| + |\frac{1}{1296(7/4)^{10}}| \approx 0.042$.

Figure 4: The graph on the left shows the areas where the denominator of $\nu(z)$ does not vanish and the values of the second image of c_2. The graph on the right are these same graphs but mapped by $1/z$. The half-plane in the right hand graph is the half-plane that we want to show is forward invariant.

Now that we have most of our information, we can continue with the mapping by finding $g(z)$ in the w-plane. To do this, we can take what we have for the $1/p(z)$ function and replace z with $w = 1/z$. This creates $g(w) = w+1+\frac{A}{w}+\theta(w)$ where $|\theta(w)| = |\nu(\sigma^{-1}(w))| \leq B|\sigma^{-1}(w)|^2 \leq \frac{B}{|w|^2}$ and B and A are the same as earlier. Figure 4 shows where all the values of the second iterate of c_2 go under the mapping of $1/z$ as well as where the circle of $|z| < 0.23$ goes under the same mapping. There is also a half-plane at $Re(w) > 4.36$. We want to show that this half-plane is forward invariant. To do this, we need to make sure that when we move the real values of w by $w \mapsto w + 1 + \frac{A}{w} + \theta(w)$, it moves forward. To do this, we need the most negative vales that A and $\theta(w)$ can take.

Recall that $|A| < 1.96$. This means that the most negative A can be is -1.96. Hence, the most negative $\frac{A}{w}$ can be is about -0.45 since $w = 4.36$. Now, we know that $|\theta(w)| \leq \frac{B}{|w|^2}$ which means that $-|\theta(w)| \geq -\frac{B}{|w|^2}$. Earlier, we also showed that $B < 4.66$ hence $-B > -4.66$. This means that since $\frac{1}{|w|} = 0.23$, the most negative $\theta(w)$ can be is -0.25. Combining all of these together, we see that $4.36 \mapsto 4.36 + 1 - 0.45 - 0.25 = 4.66$. This leads us to the following proposition:

Proposition 3.7. *Under iteration of* $w \mapsto w + 1 + \frac{A}{w} + \theta(w)$, *every point* w *in the right half-plane* $Re(w) > 4.36$ *converges to infinity: the real parts of points in its orbit converge to* $+\infty$.

Proof. We will show this using induction. The first step has already been shown above. If we let $w_1 = 4.36$, we will start with the base case $n = 1$. Following the same process as above, we found that $w_1 \mapsto 4.66 = w_2$. Hence, $w_1 < w_2$ and so the base case is true.

Now, we want to assume that n holds. This means that $w_{n-1} \mapsto w_n$ and hence $w_{n-1} < w_n$. So, we need to look at where w_n gets mapped to: $w_n \mapsto w_n + 1 + A/w_n + \theta(w_n)$. Since we know that $w_n > w_{n-1}$, we can say that $1/w_n < 1/w_{n-1}$. So, we have that $A/w_n < A/w_{n-1}$ which means that the most negative it can be is $-A/w_{n-1}$. Now, we need to look at a bound on $\theta(w_n)$. By definition, $|\theta(w_n)| \leq B/|w_n|^2 < B/|w_{n-1}|^2$ which means that $-|\theta(w_n)| > -B/|w_{n-1}|^2$. So, with these replacements, we see that $w_n \mapsto w_n + 1 + A/w_n + \theta(w_n) > w_n + 1 - A/w_{n-1} - B/|w_{n-1}|^2$. However, since $w_n > w_{n-1}$, we know that $w_{n-1} + 1 - A/w_{n-1} - B/|w_{n-1}|^2 < w_n + 1 - A/w_n - B/|w_{n-1}|^2$. We also know that $w_{n-1} + 1 - A/w_{n-1} - B/|w_{n-1}|^2 = w_n$. So, if we let $w_{n+1} = w_n + 1 + A/w_n + \theta(w_n) > w_n + 1 - A/w_{n-1} - B/|w_{n-1}|^2 > w_{n-1} + 1 - A/w_{n-1} - B/|w_{n-1}|^2 = w_n$, we have that $w_n < w_{n+1}$ which completes the proof. ☐

What this proposition has told us is that the real values in the half-plane in the w-plane converges to infinity. Since this line is mapped backwards by $1/z$, when this line converges to infinity, we have that the resulting circle converges to the origin. To see this, consider the half-planes given by $x > 4.36$ and $x > 4.66$. Using the method explained in section 1, we can see that these two lines map to these two circles, respectively: $(x - 0.115)^2 + y^2 < 0.013$ and $(x - 0.107)^2 + y^2 < 0.012$. When these two circles are graphed together, we see that the second is mapped inside the first but both are tangent to the origin. Hence, as the real value of the half-plane converges to infinity, these circles converge to the origin. This means that since this area that represents all values of the second iterate of c_2 within our annulus is contained within the half-plane, we can see that the second iterate c_2 must converge to the origin. Hence, c_2 must converge to the origin as well and this completes our proof of Corollary 3.6. ☐

Combining the proofs of Corollaries 3.5 and 3.6 as well as the proof for Proposition 3.7, we can see that Lemma 3.4 has been proven.

☐

Since we were able to show that c_2 converges to the origin within our annulus, we can say something about c_1 in the other region under consideration. Because $f_c(z) = f_{1/c}(z)$, we can say that $f_c(c_2(c)) \Leftrightarrow f_{1/c}(c_2(c)) \Leftrightarrow f_{1/c}(c_1(1/c))$. Hence, within the other area (since it is where c_2 is mapped by $1/c$) we can say that c_1 converges to the origin within region b. With these new pieces of information, we can look back to the charts for regions b and d:

Region b	c_1	c_2	Region d	c_1	c_2
SMVC	y	n	SMVC	n	y
$\to 0$	y	?	$\to 0$	?	y

As you can see, we can now say that c_1 converges to the origin in region b and that c_2 converges to the origin in region d which completes the pattern that we were looking for. Hence, we know that for any cubic, there is a critical point satisfying the SMVC bound and converging to the origin.

4 Concluding Remarks

This paper has shown that Complex Dynamics can help give an idea of which critical points satisfy SMVC for quadratic and cubic polynomials. It began with a brief introduction the problem as well as some basic background information in Complex Dynamics and Complex Analysis. From there, it described the quadratic case as an example for the cubic case. In the cubic case, we showed that the intuition that convergence of a critical point and SMVC do have a relation by using a specific instance of the Petal Theorem. From here, if we can show this intuition is true for higher order polynomials, we may have a different possibility for proving that this form of SMVC is true. There is also the possibility of looking into the following stronger form of SMVC:

Conjecture 4.1. *Let $f(z) = z + \sum_{i=2}^{n} a_i z^i$ of degree $n \geq 2$ be a complex valued polynomial for which $f(0) = 0$ and $f'(0) = 1$. Then, there exists a critical point c of f that satisfies: $|\frac{f(c)}{c}| \leq \frac{d-1}{d}$ where d is the degree of the polynomial.*

This has been shown to be true for polynomials of degree 2, 3, and 4 by algebraic methods in a paper by Q. Rahman and G. Schmeisser [4]. However, the link between this version of SMVC and the convergence of the orbits of the critical points does not seem to have been looked into.

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